Pi Day Challenge Answers

Problem 1. In total, the Einsteins ate

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{19}{20}$$

of the pie. Therefore, $1-19/20=1/20$ of the pie is left. Since the radius is $20/2=10$ inches, the area of the pie is $\pi(10^2)=100\pi$. The amount of pie remaining is thus

$$\frac{1}{20}(100\pi)=\boxed{5\pi}\ \text{in}^2.$$

Problem 2. The problem is really asking: In how many ways can we arrange 10 items? If the items were all different, there would be 10! ways. However, some of the digits are the same. If we have two 3's as the second and fourth digits, for example, it wouldn't matter which of them was at second and which was at fourth. Each time a digit appears twice, we have overcounted it 2! times or twice. The digits, 1, 3, and 5 appear twice, so the number of distinct anagrams is

$$\frac{10!}{(2!)(2!)(2!)}=\boxed{453600}.$$

Problem 3. A year that is not a leap year has 365 days. When 365 is divided by 7, the remainder is 1. This means that a day in the next year is one day further into the week than the previous year. For example, January 1, 2005 fell on a Saturday, while January 1, 2006 fell on a Sunday. However, a leap year has 366 days, producing a remainder of 2 when divided by 7. Thus, a day in the next year is two days further into the week. For example, January 1, 2008 fell on a Tuesday, but January 1, 2009 will fall on a Thursday rather than a Wednesday.

A leap year can be thought of as an addition of one day to a normal year. With this in mind, we can then first count the total number of years passed, and then the number of leap years. 302 years have passed from 1706 to 2008. Out of those years, 1708, 1712, . . . , 2004, 2008 are divisible by 4. The number of years divisible by 4 in that range is $\frac{2008-1708}{4}+1=76$. However, on the Gregorian calendar, a year that is divisible by 100 is *not* a leap year, unless it is divisible by 400, in which case it *is*. Thus, we must subtract out 1800 and 1900 (keeping 2000), so $76-2=74$. In terms of days of the week, we have "advanced" $302+74=376$ days, whose remainder is 5 when divided by 7. Going back 5 days from Friday, we have March 14, 1706 on a $\boxed{\text{Sunday}}$.

Problem 4. There are many ways to solve this problem; the following is just one solution. Note that $3\cdot 36^\circ=2\cdot 54^\circ=108^\circ$. We can thus write $\cos(3\cdot 36^\circ)=\cos(2\cdot 54^\circ)$. Using the double- and triple-angle formulas, $4\cos^3 36^\circ-3\cos 36^\circ=1-2\sin^2 54^\circ$. Noting that $\sin 54^\circ=\cos 36^\circ$, this equation becomes $4\cos^3 36^\circ-3\cos 36^\circ=1-2\cos^2 36^\circ$.

Letting $x=\cos 36^\circ$,

$$4x^3-3x=1-2x^2$$

$$4x^3+2x^2-3x-1=0$$

$$(x+1)(4x^2-2x-1)=0.$$

Obviously $\cos 36° \neq -1$, so $4x^2 - 2x - 1 = 0$. Using the quadratic equation, we obtain $x = \dfrac{1 \pm \sqrt{5}}{4}$. Since $\cos 36° > 0$, we take the positive root, $\boxed{\dfrac{\sqrt{5}+1}{4}}$.

Problem 5. Note that $-\sin\dfrac{\pi}{7} = \sin\left(\pi + \dfrac{\pi}{7}\right) = \sin\dfrac{8\pi}{7}$. Then the requested value is $\sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7}$. $\sin\dfrac{2k\pi}{7}$ can be thought of as the imaginary part of $\left(\cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}\right)^k$.

Define ω_7 to be the primitive seventh root of unity; i.e., $\omega_7^7 = 1$ and $\omega_7 = \cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$. $\sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7}$ is then the imaginary part of a number which we shall call ω, such that $\omega = \omega_7 + \omega_7^2 + \omega_7^4$. We have

$$\begin{aligned}
\omega^2 &= (\omega_7 + \omega_7^2 + \omega_7^4)^2 \\
&= \omega_7^2 + \omega_7^4 + \omega_7^8 + 2\omega_7^3 + 2\omega_7^5 + 2\omega_7^6 \\
&= \omega_7 + \omega_7^2 + 2\omega_7^3 + \omega_7^4 + 2\omega_7^5 + 2\omega_7^6. \\
\omega^2 + \omega &= 2\omega_7 + 2\omega_7^2 + 2\omega_7^3 + 2\omega_7^4 + 2\omega_7^5 + 2\omega_7^6 \\
&= 2[(1 + \omega_7 + \omega_7^2 + \omega_7^3 + \omega_7^4 + \omega_7^5 + \omega_7^6) - 1] \\
&= 2(0-1) \\
&= -2.
\end{aligned}$$

Solving $\omega^2 + \omega + 2 = 0$, we obtain $\omega = \dfrac{-1 \pm i\sqrt{7}}{2}$. Since $\sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7} > 0$, we take the positive imaginary part, $\boxed{\dfrac{\sqrt{7}}{2}}$.