

### 3<sup>rd</sup> Annual Piathlon

**Directions:** Each multiple-choice problem is worth **5 points**. You will receive 5 points for each correct answer, 1 point for each problem left unanswered, and 0 points for each incorrect answer. The problems are arranged by category, and within each category, by difficulty. The categories are general mathematics (#1–13), general physics (#14–15), Galileo’s life and times (#16–20), Jupiter and its moons (#21–25), and general astronomy (#26–30). These are intended to be challenge problems, so don’t be discouraged if you can only solve a few. Please submit your Piathlon Answer Form to Mrs. Grasel in 43D or to your math teacher by **Wednesday, March 10, 2010**. Prizes will be awarded to top scorers. A separate prize will be awarded for each grade level to offset any advantage older students may have.

**Warning:** Do not cheat! Teachers have been instructed not to give extra credit for the multiple-choice section. As for the free-response section, it is quite obvious if the solutions submitted by several students are essentially identical. You are only helping your competitors by sharing your answers with them (after all, this is a contest with prizes).

1. A pitcher fully filled with water weighs 2010 grams. Half filled, it weighs 1610 grams. How many grams does an empty pitcher weigh?  
(A) 1010 (B) 1210 (C) 1410 (D) 1810 (E) 2410
2. Two sides of a triangle are 7 and 14. Which of the following is a possible length of the third side?  
(A) 2 (B) 5 (C) 7 (D) 14 (E) 21
3. A total of 50 students attend Io High School. 30 of them are taking physics, 20 are taking astronomy, and 10 are taking both. How many are taking neither?  
(A) 0 (B) 10 (C) 20 (D) 30 (E) 40
4.  $(\sec^2 \theta - 1)(\sin^2 \theta - 1) =$   
(A)  $\sin^2 \theta$  (B)  $-\sin^2 \theta$  (C)  $\cos^2 \theta$  (D)  $-\cos^2 \theta$  (E)  $\tan^2 \theta$
5. For a pie with diameter  $d$  inches, the area in square inches is numerically equivalent to the circumference in inches. Find  $d$ .  
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5
6. Working together, Alice and Bob can make a pie in 3 hours. Working together, Alice and Charlotte can make a pie in 4 hours. Working together, Bob and Charlotte can make a pie in 6 hours. How many minutes does it take for all three people working together to make a pie?  
(A) 120 (B) 130 (C) 140 (D) 150 (E) 160

7. A cube-shaped box of side length 1 is placed on a table. The top is removed, and a sphere of radius 1 is fitted snugly on top of the open box. Find the distance from the center of the sphere to the table.  
(A)  $\frac{10}{7}$  (B)  $1 + \frac{\sqrt{2}}{2}$  (C)  $1 + \frac{\sqrt{3}}{2}$  (D)  $1 + \frac{2\sqrt{2}}{3}$  (E) 2
8. The inhabitants of Ganymede use a type of currency known as quarks. Quarks are issued in only two denominations: five-quark pieces and eight-quark pieces. The largest monetary value that cannot be represented using any combination of the two types of coins is  $x$  quarks. Find the sum of the digits of  $x$ .  
(A) 9 (B) 11 (C) 13 (D) 15 (E) 17
9. In how many ways can the letters in APPLEPIE be arranged?  
(A) 120 (B) 3360 (C) 6720 (D) 20160 (E) 40320
10. What is the last digit of  $2^{50}$ ?  
(A) 0 (B) 2 (C) 4 (D) 6 (E) 8
11. Two points,  $A$  and  $B$ , are given in the Euclidean plane. Then the locus of all points  $C$  such that  $AC = k \cdot BC$ , where  $k > 1$ , consists of  
(A) two intersecting lines  
(B) a circle  
(C) a non-circular ellipse  
(D) a parabola  
(E) a hyperbola
12. As everyone knows, February 14 is Pi Minus One Day. To celebrate, Prof. Euler decides to make a cardioid (heart-shaped) pie with area  $\pi - 1$ . If the polar equation of the cardioid is  $r = a + a \cos \theta$ , find  $a$ .  
(A)  $\sqrt{\frac{\pi - 1}{3\pi}}$  (B)  $\sqrt{\frac{\pi - 1}{2\pi}}$  (C)  $\sqrt{\frac{2(\pi - 1)}{3\pi}}$  (D)  $\sqrt{\frac{\pi - 1}{\pi}}$  (E)  $\sqrt{\frac{2(\pi - 1)}{\pi}}$
13. In biology, the differential equation for logistic growth is given by  $dP/dt = rP(K - P)$ , where  $P$  is the current population,  $t$  is time in years,  $r$  is a constant affecting the growth rate, and  $K$  is the carrying capacity. A population of *G. galilei* obeys this equation and has a carrying capacity of 1000 and an  $r$ -value of 0.001. If there are initially 10 organisms, what will the population be in 5 years?  
(A) 300 (B) 400 (C) 500 (D) 600 (E) 700

14. Two one-inch diameter iron balls are placed at opposite ends of a 1 in  $\times$  1 in  $\times$  10 in box. Two two-inch diameter iron balls are placed at opposite ends of a 2 in  $\times$  2 in  $\times$  10 in box. The ratio between the gravitational attraction of the two-inch balls and the gravitational attraction of the one-inch balls is  
(A) 4 (B) 9 (C) 36 (D) 64 (E) 81
15. Two stationary charges  $+Q$  are located on the x-axis at  $(L, 0)$  and  $(-L, 0)$ . Another charge  $+q$  ( $q \ll Q$ ), initially placed at  $(0, 0)$ , is restrained such that it is free to move horizontally but not vertically. It is displaced from this position by a small distance  $x$  ( $x \ll L$ ), and proceeds to oscillate between  $(x, 0)$  and  $(-x, 0)$ . Find the period of oscillation.  
(A)  $\sqrt{\frac{\pi^3 \epsilon_0 L^3}{4Qq}}$  (B)  $\sqrt{\frac{\pi^3 \epsilon_0 L^3}{2Qq}}$  (C)  $\sqrt{\frac{\pi^3 \epsilon_0 L^3}{Qq}}$  (D)  $\sqrt{\frac{2\pi^3 \epsilon_0 L^3}{Qq}}$  (E)  $\sqrt{\frac{4\pi^3 \epsilon_0 L^3}{Qq}}$
16. Galileo was born on the day that the famous Renaissance artist \_\_\_\_\_ died.  
(A) Botticelli (B) Donatello (C) Leonardo (D) Michelangelo (E) Raphael
17. Galileo was tried by the Inquisition for his work  
(A) *The Assayer*.  
(B) *Dialogue Concerning the Two Chief World Systems*.  
(C) *Discourse Concerning the Natation of Bodies*.  
(D) *Discourses and Mathematical Demonstrations Relating to Two New Sciences*.  
(E) *The Starry Messenger*.
18. Galileo discovered that the planet \_\_\_\_\_ had different phases, just like the moon.  
(A) Mercury (B) Venus (C) Mars (D) Jupiter (E) Saturn
19. Who extolled Galileo as “the greatest mind of all time”?  
(A) Descartes (B) Einstein (C) Grotius (D) Lucretius (E) Newton
20. At the University of Pisa, Galileo initially studied  
(A) mathematics and philosophy.  
(B) physics and philosophy.  
(C) medicine and philosophy.  
(D) mathematics and medicine.  
(E) physics and theology.

21. The large anti-cyclonic storm in Jupiter's atmosphere is known as the \_\_\_\_\_ Red Spot.  
(A) Gargantuan (B) Giant (C) Gigantic (D) Grand (E) Great
22. What is Jupiter's largest moon by mass?  
(A) Amalthea (B) Callisto (C) Europa (D) Ganymede (E) Io
23. By average radius, which moon is closest to Jupiter?  
(A) Adrastea (B) Amalthea (C) Io (D) Metis (E) Thebe
24. In which year was the greatest number of Jovian moons discovered?  
(A) 2003 (B) 2004 (C) 2005 (D) 2006 (E) 2007
25. Which Jovian moon's orbit has the largest eccentricity (i.e. deviates the most from a circular orbit)?  
(A) Aoede (B) Chaldene (C) Erinome (D) Himalia (E) Leda
26. Hydrogen makes up approximately \_\_\_\_% of the total mass of the universe.  
(A) 55 (B) 65 (C) 75 (D) 85 (E) 95
27. Which dwarf planet in the Solar System has the greatest mass?  
(A) Ceres (B) Eris (C) Haumea (D) Makemake (E) Pluto
28. In general, the heaviest element that can be formed outside of a supernova is  
(A) neon. (B) sodium. (C) iron. (D) gold. (E) lead.
29. Who first discovered a comet not visible to the naked eye?  
(A) Galileo (B) Halley (C) Herschel (D) Huggins (E) Mitchell
30. Which of the following is not a main sequence star?  
(A) Alpha Centauri (B) Antares (C) Barnard's Star (D) Fomalhaut (E) Sun

$\pi \quad \pi \quad \pi \quad \pi \quad \pi \quad \pi \quad \pi \quad \pi \quad \pi \quad \pi \quad \pi \quad \pi \quad \pi$

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**Free-Response Questions:** For the following questions, you will need to show all your work with complete justification in order to receive full credit. Each problem is worth 20 points.

1.  $ABC$  is an equilateral triangle. Point  $P$  lies on arc  $BC$  of the circumcircle of  $\triangle ABC$ , and  $\overline{AP}$  intersects  $\overline{BC}$  at point  $Q$ . Prove that  $\frac{1}{PB} + \frac{1}{PC} = \frac{1}{PQ}$ . (Hint: Use similar triangles.)
2. Prove that for any positive integer  $n$ ,  $(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$ . (Hint: Use induction.)
3. Find  $\sin \frac{\pi}{9} \sin \frac{2\pi}{9} \sin \frac{3\pi}{9} \sin \frac{4\pi}{9}$  without a calculator. (Hint: Use the formula  $\sin \theta = \frac{z - z^{-1}}{2i}$ , where  $z = \cos \theta - i \sin \theta$ .)
4. Two identical twins are standing on top of a ramp with angle of elevation  $\theta$ . They are holding identical pies in their hands, which can be modeled as solid cylinders with mass  $M$  and radius  $R$ . One of the twins holds his pie such that its bottom is touching the ramp. The other holds his pie such that its rim is touching the ramp. They both let go of the pies at the same time. If the pies arrive at the bottom of the ramp at the same time, find the coefficient of kinetic friction  $\mu_k$  in terms of  $\theta$ . (Assume that the second pie rolls without slipping.)
5. A battery with potential  $V$  and two resistors with resistance  $R$  and  $r$  are connected in series. Find the value of  $r$  in terms of  $R$  that maximizes the power dissipated across the resistor with resistance  $r$ .

Sponsored by WHS Math Club. Written and typeset in L<sup>A</sup>T<sub>E</sub>X by Tony Jin.

$$\lim_{x \rightarrow \infty} \text{math} = \pi$$

The Pi's the Limit!